

1 volume conservation

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'} + \frac{\partial f}{\partial z'} \frac{\partial z'}{\partial x} \quad (1)$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial y'} + \frac{\partial f}{\partial z'} \frac{\partial z'}{\partial y} \quad (2)$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial z'} \frac{\partial z'}{\partial z} \quad (3)$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial t'} + \frac{\partial f}{\partial z'} \frac{\partial z'}{\partial t} \quad (4)$$

for $f = z$, one obtains:

$$\frac{\partial z}{\partial x'} = -J \frac{\partial z'}{\partial x} \quad (5)$$

$$\frac{\partial z}{\partial y'} = -J \frac{\partial z'}{\partial y} \quad (6)$$

$$\frac{\partial z}{\partial t'} = -J \frac{\partial z'}{\partial t} \quad (7)$$

The volume conservations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (8)$$

$$\frac{\partial u}{\partial x'} + \frac{\partial u}{\partial z'} \frac{\partial z'}{\partial x} + \frac{\partial v}{\partial y'} + \frac{\partial v}{\partial z'} \frac{\partial z'}{\partial y} + \frac{\partial w}{\partial z'} \frac{\partial z'}{\partial z} = 0 \quad (9)$$

$$\textcolor{red}{J} \frac{\partial u}{\partial x'} + \textcolor{red}{J} \frac{\partial z'}{\partial x} \frac{\partial u}{\partial z'} + \textcolor{red}{J} \frac{\partial v}{\partial y'} + \textcolor{red}{J} \frac{\partial z'}{\partial y} \frac{\partial v}{\partial z'} + \frac{\partial w}{\partial z'} = 0 \quad (10)$$

$$J \frac{\partial u}{\partial x'} - \frac{\partial z}{\partial x'} \frac{\partial u}{\partial z'} + J \frac{\partial v}{\partial y'} - \frac{\partial z}{\partial y'} \frac{\partial v}{\partial z'} + \frac{\partial w}{\partial z'} = 0 \quad (11)$$

since

$$\frac{\partial z}{\partial x'} \frac{\partial u}{\partial z'} = \frac{\partial}{\partial z'} \left(\frac{\partial z}{\partial x'} u \right) - \frac{\partial J}{\partial x'} u \quad (12)$$

$$\frac{\partial z}{\partial y'} \frac{\partial v}{\partial z'} = \frac{\partial}{\partial z'} \left(\frac{\partial z}{\partial y'} v \right) - \frac{\partial J}{\partial y'} v \quad (13)$$

$$J \frac{\partial u}{\partial x'} + \frac{\partial J}{\partial x'} u + J \frac{\partial v}{\partial y'} + \frac{\partial J}{\partial y'} v + \frac{\partial w}{\partial z'} - \frac{\partial}{\partial z'} \left(u \frac{\partial z}{\partial x'} \right) - \frac{\partial}{\partial z'} \left(v \frac{\partial z}{\partial y'} \right) = 0 \quad (14)$$

$$\frac{\partial Ju}{\partial x'} + \frac{\partial Jv}{\partial y'} + \frac{\partial}{\partial z'} \left(w - u \frac{\partial z}{\partial x'} - v \frac{\partial z}{\partial y'} \right) = 0 \quad (15)$$

The definition of ω

$$\omega = \frac{\partial z'}{\partial t} + u \frac{\partial z'}{\partial x} + v \frac{\partial z'}{\partial y} + w \frac{\partial z'}{\partial z} \quad (16)$$

$$w = J\omega - J \frac{\partial z'}{\partial t} - Ju \frac{\partial z'}{\partial x} - Jv \frac{\partial z'}{\partial y} \quad (17)$$

$$= J\omega + \frac{\partial z}{\partial t'} + u \frac{\partial z}{\partial x'} + v \frac{\partial z}{\partial y'} \quad (18)$$

substituting w

$$\frac{\partial Ju}{\partial x'} + \frac{\partial Jv}{\partial y'} + \frac{\partial}{\partial z'} \left(J\omega + \frac{\partial z}{\partial t'} \right) = 0 \quad (19)$$

finally:

$$\frac{\partial J}{\partial t'} + \frac{\partial Ju}{\partial x'} + \frac{\partial Jv}{\partial y'} + \frac{\partial J\omega}{\partial z'} = 0 \quad (20)$$