Local ensemble assimilation scheme with global constraints and conservation

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Covariance localization

Need for covariance localization

- In ensemble assimilation schemes, the model error covariance $\mathbf{P}$ is represented by an **ensemble of model states** $\mathbf{x}^{(k)}, k = 1, \ldots, N$ ($\langle \dot{} \rangle$ is the ensemble average).

$$\mathbf{P} = \langle (\mathbf{x} - \langle \mathbf{x} \rangle)(\mathbf{x} - \langle \mathbf{x} \rangle)^T \rangle = \mathbf{X}\mathbf{X}^T$$

- As N increases **convergence is relatively slow** $(N^{-1/2}) \rightarrow$ sampling error.
- This sampling error leads to **unrealistic long-range correlations**.
- Covariance localization suppresses these long-range correlations based on the horizontal distance based on a **specified length-scale**.

Global constraints and conservation

- Localization splits the assimilation problem into a series of local optimizations
- Global assimilation schemes have no problem in respecting linear conservation
- Non-linear constraints can sometimes be transformed into linear constraints by a careful transformation model variable. Example
  - Layered models: salinity S_i and layer thickness h_i, then :

$$\sum_i \int_\Omega S_i h_i \, \mathrm{d}\mathbf{x} = \text{const}$$

  - This conservation property is non-linear if a state vector including S_i and h_i
  - ... but becomes linear if the state vector includes $S_i h_i$ and S_i (or h_i).

Localization

- One can distinguish different localization approaches:
  - **covariance localization**: every single observation point is assimilated sequentially and the correction are filtered by a localization function. (less suited for parallel processing and the domain localization).
  - **domain localization**: the state vector is decomposed into sub-domains (e.g. single grid points or vertical columns) where the assimilation is performed independently. Such algorithm are easily applied to parallel computers.
- Conservation of the global property is lost if the assimilation is performed locally
- The conservation requires a coupling of a model grid points which is filtered-out by the localization.
- Similar difficulty: non-local observation operator

Method

- We propose a assimilation scheme which is local and can satisfy global conservation properties and non-local observation operators.
- In essence:
  - Based on **covariance localization**
  - Localize ensemble covariance matrix (by using an element-wise matrix product)
  - Modify this localized covariance matrix to so that the **uncertainty of the total amount of the conserved quantity is zero**
  - One recovers the original Kalman filter analysis if the covariance does not have spurious long-range correlation.
  - Parallel algorithm

Solver

- Matrices are not formed explicitly, but as an "operator"

$$\mathbf{P}_c = (\mathbf{I} - \mathbf{h}\mathbf{h}^T)(\boldsymbol{\rho} \circ \mathbf{P})(\mathbf{I} - \mathbf{h}\mathbf{h}^T) \tag{1}$$

  where $\mathbf{h}^T(\mathbf{x}^a - \mathbf{x}^f) = 0$

- Conjugate gradient algorithm as solver for these systems:

$$\left(\mathbf{H}\mathbf{P}_c\mathbf{H}^T + \mathbf{R}\right)^{-1}\mathbf{y} = \mathbf{b} \tag{2}$$

  where $\mathbf{y}$ and $\mathbf{b}$ two vectors in the observation space.

- Preconditioner can be:
  - Solution without localization $\mathbf{P}_c \sim \mathbf{P}$
  - Solution without ensemble (3D-Var) $\mathbf{P}_c \sim \boldsymbol{\rho}$

Variants

- Ensemble mean: standard Kalman Filter update (but with modified error covariance)

$$\mathbf{x}^a = \mathbf{x}^f + \mathbf{K}(\mathbf{y}^o - \mathbf{H}\mathbf{x}^f) \tag{3}$$

- Full ensemble:
  - Either use perturbed observations $\mathbf{y}^{o(k)}$ (variant "pert")

$$\mathbf{x}^{a(k)} = \mathbf{x}^{f(k)} + \mathbf{K}(\mathbf{y}^{o(k)} - \mathbf{H}\mathbf{x}^{f(k)}) \tag{4}$$

  - Project analysis error covariance onto a subspace (variant $\mathbf{P}_c$)
  - Project approximated analysis error covariance onto a subspace (variant $\mathbf{S}\mathbf{S}^T$). For this variant: no rotation if $\mathbf{R} \to \infty$

Test case

Kuramoto-Sivashinsky equation

- Equations:

$$\partial_t v = -\partial_x^2 v - \partial_x^4 v - v\partial_x v \tag{5}$$

- Periodic domain: $L = 32\pi$ with 128 grid points
- Time-step: $\Delta t = 1/4$
- ETDRK4 (Exponential Time Differencing fourth-order Runge-Kutta)
- Conservation:

$$\frac{d}{dt}\int_0^L v \, dx = 0 \tag{6}$$

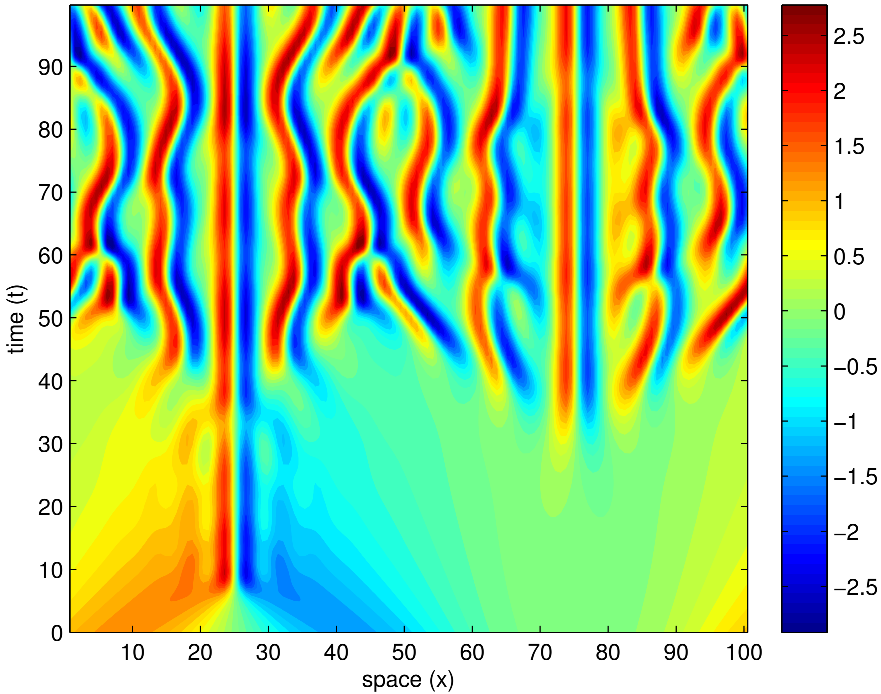


Figure 1: Solution of the KS equation (without assimilation)

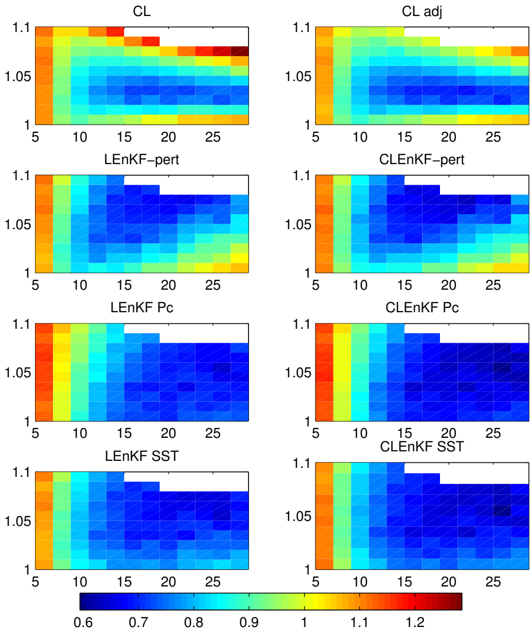
Assimilation test cases:

	non-conservative	conservative
covariance localization	CL	CL-adj
perturbed observation	EnKF-pert	CEnKF-pert
Localized EnKF without perturbed obs. variant "$\mathbf{P}_c$"	LEnKF-$\mathbf{P}_c$	CLEnKF-$\mathbf{P}_c$
Localized EnKF without perturbed obs. variant "$\mathbf{SS}^T$"	LEnKF-$\mathbf{SS}^T$	CLEnKF-$\mathbf{SS}^T$

Assimilation setup

- Classical twin experiment
- Every 8th grid point is observed (with an error variance of 0.1) at every 10 model time steps
- The model with assimilation for 1000 time steps
- The experiment is repeated 1000 times and RMS errors relative to the true solution are averaged.
- Using different localization length-scale and inflation factors

Results



- RMS error between the model run with assimilation and true solution for different schemes
- x-axis: localization length-scale
- y-axis: inflation factors
- white region where model is unstable

Optimal parameters

	L	inflation	mean RMS	std of mean RMS
CL	21	1.03	0.71375	0.00271
CL adj	21	1.03	0.68624	0.00268
LEnKF-pert	21	1.07	0.66267	0.00570
CLEnKF-pert	21	1.07	0.63493	0.00609
LEnKF $\mathbf{P}_c$	25	1.05	0.64253	0.00364
CLEnKF $\mathbf{P}_c$	25	1.05	**0.59395**	0.00386
LEnKF $\mathbf{SS}^T$	25	1.05	0.64078	0.00513
CLEnKF $\mathbf{SS}^T$	25	1.05	0.59953	0.00452

- Lowest RMS for different assimilation schemes and corresponding parameters
- Methods with conservation always better than without
- CLEnKF $\mathbf{P}_c$ and CLEnKF $\mathbf{SS}_c$ very similar, but CLEnKF $\mathbf{P}_c$ slightly better

Minimal model for sea ice and salinity with conservation

- Assess these schemes for a multivariate model
- Minimal model for sea ice and salinity where the amount of "freshwater" (or salt) is conserved.
- Integral of a function f (of the model parameter) over a closed domain remains constant over time:

$$\frac{d}{dt}\int_{\Omega} f\,dx = 0 \tag{7}$$

- The velocity (v) for salinity (S) is provided using the Kuramoto-Sivashinsky equation:

$$\partial_t v = -\partial_x^2 v - \partial_x^4 v - v\partial_x v - g\partial_x h \tag{8}$$

Governing equations

- The flow v is not "incompressible" as it varies with x. Thus we use also the variable h, representing the height of the mixed layer.

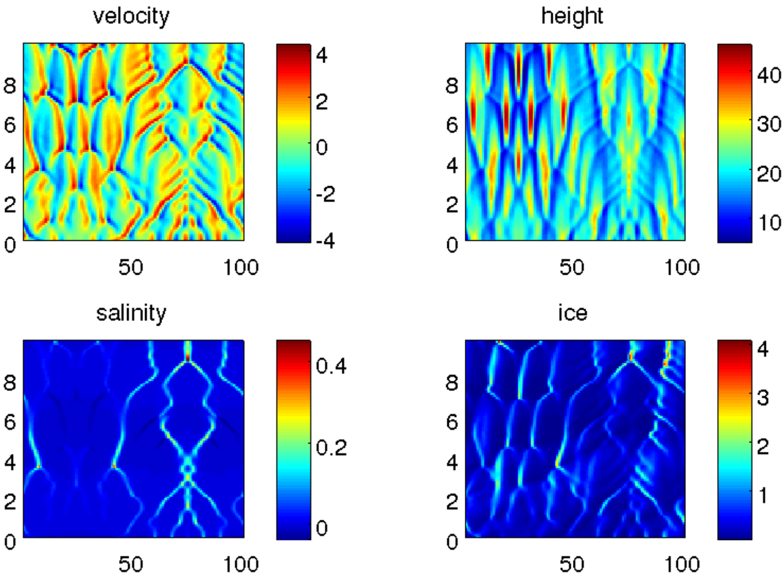
$$\begin{aligned} \partial_t(hS) + \partial_x(vhS) &= \kappa\partial_x^2(hS) + \mu\mathcal{F} \\ \partial_t c + \partial_x((v_c + v)c) &= \mathcal{F} \end{aligned}$$

  where v_c is the velocity of the sea ice (constant) and h is governed by:

$$\partial_t h + \partial_x(hv) = 0$$

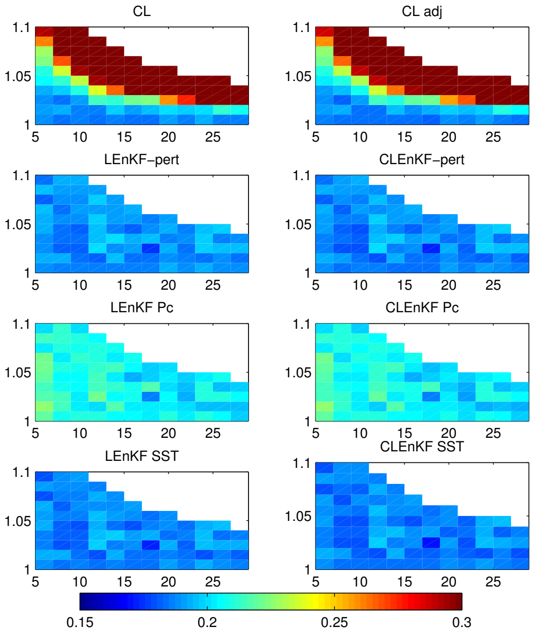
- For a periodic domain Ω, salinity fluxes and ice fluxes cancel after integration over the whole domain and one obtains:

$$\frac{d}{dt}\int_\Omega (hS - \mu c)\, dx = 0$$



- Free running simulation of the coupled multivariate model
- Solution is strongly dominated by the chaotic behavior of the velocity equation

Results



- Every second ice grid point is observed
- Average RMS error between the model run with assimilation and the true solution for different schemes and parameters
- High inflation values lead to unrealistic results for “CL” and “CL-adj”
- Results appear noisy but even after increasing the number of experiments, these small-scale variations remained and were stable

Summary

	L	inflation	mean RMS	std of mean RMS
CL	17	1.00	0.18362	0.00070
CL adj	7	1.02	0.18228	0.00047
LEnKF-pert	17	1.02	0.17444	0.00063
CLEnKF-pert	17	1.02	0.17254	0.00064
LEnKF $\mathbf{P}_c$	17	1.02	0.18689	0.00080
CLEnKF $\mathbf{P}_c$	17	1.02	0.18549	0.00080
LEnKF $\mathbf{SS}^T$	17	1.02	0.17244	0.00064
CLEnKF $\mathbf{SS}^T$	17	1.02	**0.17064**	0.00065

Table 1: Lowest RMS for different assimilation schemes and corresponding parameters

- Suprisingly $\mathbf{P}_c$ is not better than CL
- Error space rotation seem to degrade results
- Best results with method CLEnKF $\mathbf{SS}^T$

Conclusions

- New assimilation scheme which is formulated globally (i.e. for the whole state vector)
  - where spurious long-range correlations can be filtered out
  - global conservation properties can be enforced
  - non-local observation operators can be used (e.g. assimilation of observation representing an average)
- Tests with Kuramoto-Sivashinsky show benefit of this approach compared to the traditional covariance localization scheme where observations are assimilated sequentially
- Even with an ad-hoc step enforcing conservation
- Beneficial also for multivariate models with conservation constraint relating different model variables
- The most consistent variant was CLEnKF $\mathbf{SS}^T$

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